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TECHNICAL NOTE NO. 556

November 1951

T E Sterne

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PROVISIONAL CRITERIA FOR RAPID INCAPACITATION BY FRAGMENTS

ABERDEEN PROVING GROUND, MD. 21005

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B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

TECHNICAL NOTE NO. 556

November 1951

PROVISIONAL CRITERIA FOR RAPID INCAPACITATION BY FRAGMENTS

Theodore E. Sterne

Project No. TB3-0112A of the Research and Development Division, Ordnance Corps

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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BALLISTIC RESEARCH LABORATORIES
TECHNICAL NOTE NO. 556

TESterne/jer
Aberdeen Proving Ground, Md.
26 November 1951

PROVISIONAL CRITERIA FOR RAPID INCAPACITATION BY FRAGMENTS

ABSTRACT

From experimental data obtained by the Biophysics Branch, Medical Laboratories, Army Chemical Center, Edgewood, Maryland, the probabilities are inferred that hits on personnel by fragments of known mass, velocity, and shape will incapacitate. The experiments were conducted against live goats, and the results were assessed by medical officer personnel. The results are classified according to three levels of incapacitation: incapacitation within 5 seconds, incapacitation within 5 minutes, and fatal or severe wounding. The inferred probabilities, based on provisionally and rapidly interpreted experimental results, are exhibited as functions of mv/A where m is the fragment mass, v the striking velocity, and A the average presented area of the fragment.

The probability results, which are subject to revision should the basic data be reinterpreted, are being currently employed by the Ballistic Research Laboratories in the design of improved hand grenades.

[redacted]

INTRODUCTION

In order to obtain experimental confirmation of the wounding power of certain newly proposed types of hand grenades the Biophysics Branch, headed by Dr. C. M. Herget, of the Medical Laboratories of the Army Chemical Center, Edgewood, Maryland, was asked by the Ordnance Corps to conduct experiments with controlled fragments against live animal targets. The experiments were planned and carried out under the direction of Dr. Herget, who was assisted by three medical officers* specially assigned by the Medical Corps to the project. The fragment sources and some experimental assistance were supplied by the Ballistic Research Laboratories. Goats were selected by Dr. Herget as the target animals. It is understood that he was governed in his choice by the desire for targets somewhat comparable in mass and dimensions to humans, and the fact that they were available in sufficient numbers without excessive delay or expense. The results of the first few experiments indicated that there was a great difference, between "fatal or severe wounding" and rapid incapacitation. Many wounds, severe and perhaps ultimately fatal, cause no such immediate changes in the target animals as to suggest that humans, in battle and under similar conditions of wounding, would have been prevented from carrying out the most important of normal battle assignments, like hurling grenades or discharging firearms. The experimental program was therefore modified and extended in an effort to obtain basic data on the probability that random hits by fragments of known mass, shape, and striking velocity would cause rapid incapacitation.

It is understood that the Biophysics Branch plans to issue full and formal reports of the experiments, which have been completed. Pending these, the Ballistic Research Laboratories have interpreted the experimental data statistically in such a way as to be able to employ the results in current design studies for improved hand grenades. Based on the experimental results as obtained informally from Dr. Herget, the inferred probabilities will be described in this report along with the method that has been employed in deriving them. The experimental data available to the writer must be regarded as provisional because they may be modified by Dr. Herget, before he publishes them, as the result of a possible critical re-examination in the light of post mortem studies by the medical officers.

The probabilities in this report should be regarded as provisional. They are published at this time only because they are being used for an extensive design study, whose urgency has rendered necessary the employment of data which, although provisional, are believed to be the best data currently available.

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* Major M. D. Blackburn, Capt. H. R. Fiege, and 1st Lt. M. Ladd.

It is a pleasure to record the enthusiastic and willing cooperation of Dr. Herget and his co-workers at Edgewood, in supporting the design studies of the Ballistic Research Laboratories, by their cheerful and energetic execution of the wound ballistic experiments.

The author bears entire responsibility for the following discussion, which it is hoped will not deprive Dr. Herget of the opportunity of re-interpreting his experimental results, or of publishing any better analysis of them that can be devised.

THE DATA

Three types of fragment sources, prepared and supplied by the Ballistic Research Laboratories, were used in the experiments. All the fragments were of steel, in the shape of rectangular parallelepipeds. The first source consisted of a sphere of high explosive, pentolite, surrounded by pre-formed fragments whose masses were all approximately 0.71 grains. The initial dimensions of all the fragments were 0.1 x 0.1 x 0.04 inches. The total number of fragments was nearly 2000, the ratio of explosive charge to total weight was about 0.69, and the total weight of the source was about 12 ounces. Recovered fragments were found to have lost, as a rule, some of their sharp edges and corners with the result that their masses were less than the original masses. Half the mass of the recovered fragments was in the form of fragments more massive than 0.60 grains, which is here called the "median" fragment mass. The initial fragment velocity was approximately 7800 ft/sec.

The second type of source consisted of a sphere of pentolite surrounded by pre-formed fragments whose initial dimensions were 0.1 x 0.1 x 0.137 inches, whose initial masses were all 2.5 grains, and of which there were approximately 1100. The median mass of recovered fragments was 2.1 grains; the ratio of explosive charge to total weight was about 0.47, and the total weight was about 12.2 ounces. The initial fragment velocity was approximately 5300 ft/sec.

The third type of fragment source was a sphere of pentolite surrounded by approximately 740 fragments, with initial masses of 5 grains and dimensions 0.125 x 0.125 x 0.164 inches. The median mass of recovered fragments was 4.26 grains; the ratio of charge to total weight was 0.39 approximately, and the total weight was nearly 14.5 ounces. The initial fragment velocity was approximately 4900 ft/sec.

A fourth source of fragments, a standard Mk II hand grenade, was also used but since the fragments were of uncertain mass and striking velocity the related experimental data will not be discussed or employed in this report.

The physical data relating to the fragment sources have been furnished to the author by Mr. Tolch and Mr. Dunn, of the Ballistic Research Laboratories, under whose supervision the sources were prepared.

Firings against the goats were at distances of 5, 10, 15, and 60 feet. At the last distance, so few hits were obtained that the resulting data will not be employed in this report. None of the few hits at 60 feet caused even severe wounds. In each experiment, four goats were at a fixed distance, either 5, 10, or 15 feet, from the fragment source. No experiment with the 2.1 grain fragments was conducted at 5 feet; some of the other experiments involving four goats were repeated with the result that the series of experiments involved eight goats under certain conditions of distance and fragment source, and 4 goats under the other conditions.

Three categories of incapacitation or wounding have been employed. The most severe category is that of incapacitation in 5 seconds or less, and is called "Type K". A less severe category is that of incapacitation in 5 minutes or less, which includes Type K. This category is called "Type A". The third and least severe category is that of fatal or severe wounding, and for reasons of logic contains Types K and A. This third category is called "Type B".

In this analysis, all experiments with the same type of fragment source at the same distance have been pooled and regarded as a single experiment, each experiment therefore involving either four or eight animals. In each experiment the fragment velocity at the distance of the animals was known from previous velocity firings or from deceleration experiments; the median mass of the fragments after firing was known; and it was assumed that the average presented area A of a fragment divided by the 2/3 power of its mass, $A/m^{2/3}$, was the same after firing as before, so that it could be evaluated by calculation from the initial mass and shape. In evaluating A, use was made of the well-established geometrical law that the average presented area of any geometrical solid, whose surface is nowhere concave, is exactly equal to 1/4 of the total superficial area.* The physical and geometrical data were therefore known, rather closely, in all the experiments. The result of an experiment, involving n animals, was that there were s instances of incapacitation or wounding of any particular type. The number n was always either 4 or 8; while s had logically to be some integer in the range from 0 through n. Actually, the medical officers were sometimes not able to state with complete assurance that there were exactly s instances, out of the n animals, of incapacitation or wounding of a particular sort. Out of eight animals, for example, they might be uncertain in a particular experiment whether one or two could truthfully be stated to have been incapacitated within 5 seconds. In such cases, the author has arbitrarily averaged the possible and doubtful values of s and has used the average, in the preceding example 1.5, as though it were the number of animals, among n fired at, observed to have been incapacitated. A final datum to be used is not biological, although it involved the animals; it is the average number of hits per animal in each experiment. This average number can be denoted by h.

Before proceeding to the analysis, it is well to indicate that the medical officers, in determining s, were trying to think of the goats

* The surface of a geometrical solid is nowhere concave if a tangent plane never intersects the solid, but only touches it, for all possible points of tangency. Thus, a rectangular parallelepiped is of this type.

as representing humans, and were not thinking of the animals as merely goats. Thus because of anatomical differences between goats and humans it was considered that certain wounds, which were serious in goats, would have been less serious in humans. Such wounds were therefore discounted to some extent by the medical officers in their interpretations of the observed effects of the fragments upon the goats. In determining whether or not an animal was "incapacitated", it is understood that the medical officers were asking themselves the hypothetical question, "If this goat has been a determined and brave soldier, would he have been able after exposure to the fragments to have employed a weapon against his enemies? Could he, for instance, have raised, aimed, and fired a pistol?" The requirements for considering a target to have been "incapacitated" were thus very severe requirements in the minds of the medical officers who classified the types of injury.

ANALYSIS AND CONCLUSIONS

If s animals are incapacitated among the n animals exposed to fragments in a particular experiment, then the fraction s/n is an unbiased estimate of the unknown true probability p_k that an animal exposed to fragments under such conditions will be incapacitated. However, although it is known from statistical theory that the fraction s/n is an unbiased estimate and is moreover the best estimate that can be obtained, it is also known that it will in general be erroneous. If the unknown true probability is p_k, then it is known that the probability of obtaining precisely s incapacitations in an experiment involving n animals is exactly

$$\frac{n!}{s!\,(n-s)!}\, p_k^{\,s}(1-p_k)^{n-s}.$$

This matter will be discussed further in the Appendix to this report but as a particular example, if the probability p_k of incapacitation is exactly .4, then in an experiment involving four animals the probabilities that exactly 0, 1, 2, 3, or 4 animals will be incapacitated are .1296, .3456, .3456, .1536, and .0256 respectively. It is clear, therefore, that from the result s of an experiment involving n animals, where n is as small as 4, the fraction s/n can differ considerably from p_k. To a lesser extent, the same remark is true for experiments involving 8 animals, and to some extent the remark is true for any experiment involving only a finite number, however large, of animals.

From the values of s and n, the unbiased estimates s/n can be obtained of the probability p_k of incapacitation, and also confidence limits. As explained in detail in the Appendix, the 50% confidence limits are such that if one always asserts that the unknown, true, probability p lies between the 50% confidence limits corresponding to the observed s one will be correct, on the average, at least 50% of the time. Similarly, the 90% confidence limits are such that if one always asserts that the unknown, true probability of incapacitation lies between the 90% limits

corresponding to the observed s then one will be correct, on the average, in at least 90% of such assertions.

In the following table, relating to "Type K" incapacitation, the first column contains the median mass of fragment in grains, the second

Table I

Type K casualties--incapacitation in 5 seconds or less

m	A/m	v	10^{-5} mv/A	h	n	s	p_k					p_{hk}				
grs	cgs	f/s	cgs				90%	50%	best	50%	90%	90%	50%	best	50%	90%
.60	1.24	5400	1.33	2.5	8	0	0	0	0	.16	.31	0	0	0	.067	.14
.60	1.24	6200	1.52	5	4	1	.026	.16	.25	.50	.68	.005	.034	.056	.13	.20
.60	1.24	7000	1.72	25	4	0	0	0	0	.30	.50	0	0	0	.014	.028
2.1	.74	4200	1.73	1.8	4	0	0	0	0	.30	.50	0	0	0	.18	.32
2.1	.74	4550	1.87	3.5	8	1.5	.04	.11	.19	.34	.50	.012	.033	.058	.11	.18
4.26	.59	4100	2.12	1.5	8	2.5	.11	.21	.31	.45	.64	.075	.15	.22	.33	.49
4.26	.59	4300	2.22	3.5	4	1	.026	.16	.25	.50	.68	.008	.049	.079	.18	.28
4.26	.59	4600	2.38	8.5	4	2	.14	.29	.50	.71	.86	.018	.040	.078	.14	.21

column the value of A/m in cgs units, the third column the remaining fragment velocity v just before impact, the fourth column the quantity 10^{-5} mv/A in cgs units, the next column the average number of hits per animal, the next column the number of animals involved in the experiment, and the next column the number of incapacitations observed. The next five columns contain the best estimate of the probability p_k of incapacitation as well as the 50% and 90% confidence limits as described by the column headings. The final five columns contain the corresponding unbiased estimates and confidence limits for the unknown probability, p_{hk}, that a random hit by a single fragment will cause Type K incapacitation. The values of p_{hk} have been inferred from the values of the probability p_k through the equation

$$p_{hk} = 1 - (1 - p_k)^{1/h}$$

which assumes that when an animal is hit h times the probability of incapacitation p_k is one minus the probability that the animal would independently survive, without incapacitation, all of h separate hits. The assumption may not be absolutely correct, but in comparison with the coarseness of the final relations, and in view of the small proportion of overlapping wounds, does not seem likely to lead to serious error.

It may be noticed that statistical theory contained in the Appendix deals only with logically possible, and therefore intergral, values of the number, s, of animals incapacitated. In the table, however, there are two instances of fractional values of s. Since both fractions are

half-integers, the confidence limits have been inferred from the tables in the Appendix by averaging the entries for the two adjacent integral values of $\underline{s}$.

In Figure 1, the probabilities p_{hk} that a random hit will cause incapacitation in 5 seconds or less are plotted against the quantity $10^{-5}mv/A$. Both the 50% and 90% confidence limits are shown. The reasons for choosing $10^{-5}mv/A$ as the physical parameter of which p_{hk} is a function are substantially the same as in BRL Technical Note No. 370*. It seems reasonable to suppose that the probability, that a random hit will incapacitate in 5 seconds or less, depends upon the fraction of the body's superficial area through which the fragment can penetrate and thereafter strike, and cause rapidly incapacitating injury to a vital region. Such vital regions, for "Type K" incapacitation, should be expected to be very much smaller than the regions which are of importance to less rapid incapacitation or to merely severe wounding. The author has been informed by the medical and biophysical staff at Edgewood that the regions vital to 5-second incapacitation are in fact very small. If fragments possess such great "penetrating power" that a fragment traverses the body completely wherever it hits, then the probability that a random hit will cause 5-second incapacitation is merely the ratio of the presented area of the relevant vital regions to the total presented body area. On the other hand, if the "penetrating power" of fragments is so low that they can never reach vital regions, wherever they hit, then the probability p_{hk} must be 0. For an intermediate degree of "penetrating power", a certain fraction only of the body's area is such that hits on it will permit penetration deep enough to reach vital regions, and of this fraction a still smaller part will correspond to actual injury of vital regions. It therefore seems not unreasonable to suppose that the probability, p_{hk}, that a random hit will incapacitate in 5 seconds or less is a function of the "penetrating power" of the fragment. It seems probable that merely reaching a vital region is not enough, and that to incapacitate rapidly the fragment must have some definite remaining energy or velocity when it reaches the vital region. This consideration cannot destroy the dependence of p_{hk} upon "penetrating power", although it can and probably does influence the nature of the dependence. It seems reasonable, therefore, after everything has been considered, to attempt to relate the probability p_{hk} to the "penetrating power" of the fragment.

The "penetrating power" of a fragment is a rather vague concept.

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* BRL Technical Note 370, "A Provisional Casualty Criterion," T. E. Sterne, March 1951.

The penetrating ability is not a property of a fragment alone, but is really a property of the fragment and of the nature of the penetrated material. The body wall differs, from place to place, in its ability to resist penetration. Thus, bone is more resistent than skin and skin in turn appears to be more resistant, thickness for thickness, than soft tissue*. Therefore the penetrating power of a fragment really depends on where it hits the human target. For the present rapid and rough analysis, penetrating power is nevertheless represented by the quantity mv/A as though the body wall were homogeneous in its properties from place to place, but varied in thickness. It is hoped that others may be able to improve the analysis in this report by taking into account the detailed anatomical variation in the resisting power of the body wall to fragments, from place to place, in some such manner as that employed by McMillen and Gregg in their report already cited.

Through the confidence intervals plotted in Figure 1, there has been drawn a rather wide curve, or band, describing the author's estimate of the relation between p_{hk} and the parameter $10^{-5}mv/A$. It seems logically necessary for p_{hk} to increase or be constant with increasing abscissa. The very small values of the upper confidence limits at abscissa 1.72** fixes the left-hand portion of the curve rather closely. This is fortunate, because the curve is not otherwise too well determined. The right-hand portion of the curve has been determined from the weighted mean of the best estimates s/n corresponding to the three greatest abscissas. It is not unreasonable for the ordinate p_{hk} to be constant for still higher values of the abscissa, because such fragments already have very great penetrating power and therefore may correspond substantially to complete traversal of the target whenever it is hit. It is interesting that the maximum value of p_{hk}, as so determined, should have been found to be approximately 0.1, for this value happens to be in exact agreement with an estimate, made by Burns, Zuckerman, and Krohn, that 10% of the body area was such as to result in death when punctured by fragments that could

* McMillen, J. H., and J. R. Gregg. The energy, mass and velocity which is required of small missiles in order to produce a casualty. Missile Casualties Report No. 12, 1945, National Research Council, Division of Medical Sciences, acting for the Committee on Medical Research, Office of Scientific Research and Development.

** Corresponding to an experiment in which there were no Type K incapacitations among 4 animals only 5 feet from .6 grain fragment source, even though the average number of hits was 25 per animal.

traverse the body completely*. This close agreement is perhaps fortuitous.

The next table contains similar data, relating to Type A incapacitation (incapacitation in 5 minutes or less). The physical conditions include the number of hits, exactly the same as in the first table,

TABLE II

Type A casualties--incapacitation in 5 minutes or less

m	A/m	v	10^{-5}mv/A	h	n	s	p_k			p_{hk}		
grs	cgs	f/s	cgs				50%	best	50%	50%	best	50%
.60	1.24	5400	1.33	2.5	8	.5	.04	.06	.22	.016	.025	.10
.60	1.24	6200	1.52	5	4	1.5	.22	.38	.60	.05	.091	.18
.60	1.24	7000	1.72	25	4	4	.70	1	1	.048	1	1
2.1	.74	4200	1.73	1.8	4	1	.16	.25	.50	.09	.15	.32
2.1	.74	4550	1.87	3.5	8	3.5	.32	.44	.56	.10	.15	.21
4.26	.59	4100	2.12	1.5	8	3	.27	.38	.51	.19	.27	.38
4.26	.59	4300	2.22	3.5	4	4	.70	1	1	.29	1	1
4.26	.59	4600	2.38	8.5	4	4	.70	1	1	.13	1	1

from which this table differs only in the numbers of incapacitated animals. 50% confidence limits are shown, and the unbiased estimates s/n, as before, but not the 90% limits which are too wide to be of any use. The resulting plot of the probability p_{hk} for "Type A" incapacitation, against the abscissa 10^{-5}mv/A, is shown in Figure 2. A rather wide region has been drawn on the graph by the author to represent the dependence of p_{hk} upon the penetration parameter mv/A. It will be noticed that this relation is less well-determined than the corresponding relation for Type K incapacitation. The reason for this is that the experiments were planned to obtain good information about Type K incapacitation, and the number of Type A incapacitations, being larger than the number of Type K incapacitations, has not provided as reliable information about p_{hk}.

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* Burns, B. D., and S. Zuckerman. The relationship between striking velocity and the damage caused to materials by a 3/32 in. steel ball. Gr. Brit., Ministry of Home Security, 1941, R. C. Report No. 232. Confidential

Table III

Type B casualties—fatal or severe wounds

m	A/m	v	$10^{-5}mv/A$	h	n	s	p_k			p_{hk}		
grs	cgs	f/s	cgs				50%	best	50%	50%	best	50%
.60	1.24	5400	1.33	2.5	8	6	.61	.75	.85	.31	.43	.53
.60	1.24	6200	1.52	5	4	4	.70	1	1	.21	1	1
.60	1.24	7000	1.72	25	4	4	.70	1	1	.048	1	1
2.1	.74	4200	1.73	1.8	4	3	.50	.75	.84	.32	.54	.64
2.1	.74	4550	1.87	3.5	8	8	.84	1	1	.41	1	1
4.26	.59	4100	2.12	1.5	8	7	.71	.875	.92	.56	.75	.81
4.26	.59	4300	2.22	3.5	4	4	.70	1	1	.29	1	1
4.26	.59	4600	2.38	8.5	4	4	.70	1	1	.13	1	1

Similar remarks apply to the third table, which relates to Type B casualties, fatal or severe wounding. The author understands that a wound was classified as "fatal or severe" if, in the opinion of the medical officers, it led to death or would have led to death in the absence of surgical attention. The plot, Figure 3, corresponding to Type B casualties leaves even wider freedom for the location, of the curve relating p_{hk} to the abscissa $10^{-5}mv/A$, than in the case of Type K or Type A incapacitations. Therefore, no attempt has been made to determine, from the data, the functional dependence of p_{hk} for fatal or severe wounds upon mv/A. Instead, the relation derived in BRL Technical Note No. 370 from the report by McMillen and Gregg, already cited, has been drawn in. The best that can be said is that the experimental data, as described by the confidence intervals and unbiased estimates, are perfectly consistent with the curve. The curve passes through six of the eight 50% confidence intervals, which is rather more of them than it should be expected to pass through by chance, and it passes through seven out of eight of the 90% confidence intervals. If the maximum of the curve were higher, it would pass through all of the 90% intervals. It would then be, perhaps, in somewhat improbably good agreement with experiment, but this consideration does not prove that the maximum should be no higher. The author thinks that it would be best to leave the curve where it is, as determined in BRL Technical Note 370, with considerable confidence that this location does not tend to exaggerate, but if anything tends to underestimate, the probability that a random hit will cause a fatal or severe wound.

The author very much hopes that Dr. Herget and his associates, at Edgewood, may be able to obtain more reliable and accurate probabilities of incapacitation by combining, with data of the type used here, the considerable knowledge that they possess of biological and medical matters. It is thought that information obtained from a detailed study of the

separate wounds inflicted during the experiments, of the areas of regions important to the various types of incapacitation relative to the total body area, and of basic anatomical and penetrability data relating to the body wall can in principle be combined. Such a combination would enable a considerably more reliable and accurate determination to be made of the important and useful probabilities, P_{hk}, than the crude determination that has been made in this report from more limited data.

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APPENDIX TO

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

TECHNICAL NOTE NO. 556

November 1951

SOME REMARKS ON CONFIDENCE OR FIDUCIAL LIMITS

Theodore E. Sterne

Project No. TB3-0112A of the Research and
Development Division, Ordnance Corps

A B E R D E E N   P R O V I N G   G R O U N D ,   M A R Y L A N D

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APPENDIX TO BALLISTIC RESEARCH LABORATORIES
TECHNICAL NOTE NO. 556

TESterne/jer
Aberdeen Proving Ground, Md.
26 November 1951

SOME REMARKS ON CONFIDENCE OR FIDUCIAL LIMITS

ABSTRACT

In a binomial problem, confidence limits of an unknown parent probability can be selected on the basis of the probability, P, of obtaining a number of successes as probable as, or less probable than, the observed number s of successes in n trials. In comparison with confidence intervals selected in accordance with procedures of Clopper and Pearson, the present intervals are narrower when s is either zero or n.

APPENDIX

If the probability of a "success" is π, then the probability that there will be precisely $\underline{r}$ successes during $\underline{n}$ trials is exactly

$$p_{n,r}(\pi) = \frac{n!}{r!\,(n-r)!}\;\pi^r(1-\pi)^{n-r}\;. \qquad (1)$$

Therefore, the probability of obtaining a number of successes as probable as or less probable than a particular number of successes $\underline{s}$ is equal to

$$P(\pi,n,s) = \sum_r p_{n,r}(\pi) \qquad (2)$$

where the summation is over all values of $\underline{r}$ for which

$$p_{n,r}(\pi) \leq p_{n,s}(\pi)\;. \qquad (3)$$

The probability, of obtaining a value of P equal to or less than some particular and possible value, P', of P is equal to P'. From the preceding considerations, confidence or fiducial limits of π can be chosen corresponding to any values of $\underline{n}$ and $\underline{s}$ and to any desired value, ϵ lying between 0 and 1, of P.

Consider the dependence of $P(\pi, n, s)$ upon the variable π, for any value of $\underline{s}$ other than 0 or $\underline{n}$. For values of $\underline{s}$ differing from 0 and $\underline{n}$ it is clear that $P(0, n, s)$ and $P(1, n, s)$ are zero, and that there are some values of π, between 0 and 1, for which $P(\pi, n, s)$ is unity. The values of π for which $P(\pi, n, s)$ is unity are values for which $p_{n,s}(\pi)$ is the largest of all terms $p_{n,r}(\pi)$. It can readily be shown that there are exactly $\underline{n}$ values of π between 0 to 1 at which $P(\pi, n, s)$ is discontinuous. Such discontinuities correspond to the "crossing" of $p_{n,s}(\pi)$ by other $p_{n,r}(\pi)$'s, and except at the $\underline{n}$ discontinuities $P(\pi, n, s)$ is a continuous function of π. When $\underline{s}$ is 0, the situation is somewhat different since then $P(0, n, 0)$ is unity while $P(1, n, 0)$ is zero. When $\underline{s}$ equals $\underline{n}$, $P(0, n, n)$ is zero while $P(1, n, n)$ is unity. Whatever may be the value of $\underline{s}$, there are just $\underline{n}$ values of π at which $P(\pi, n, s)$ is discontinuous.

The preceding description of some of the properties of the P-function has been included to clarify, in the reader's mind, the nature of the dependence of $P(\pi, n, s)$ upon π. To select confidence or fiducial limits,

$$p_1 \leq \pi \leq p_2$$

APPENDIX

If the probability of a "success" is π, then the probability that there will be precisely $\underline{r}$ successes during $\underline{n}$ trials is exactly

$$p_{n,r}(\pi) = \frac{n!}{r!\,(n-r)!}\,\pi^r(1-\pi)^{n-r} . \qquad (1)$$

Therefore, the probability of obtaining a number of successes as probable as or less probable than a particular number of successes $\underline{s}$ is equal to

$$P(\pi,n,s) = \sum_r p_{n,r}(\pi) \qquad (2)$$

where the summation is over all values of $\underline{r}$ for which

$$p_{n,r}(\pi) \leq p_{n,s}(\pi) . \qquad (3)$$

The probability, of obtaining a value of P equal to or less than some particular and possible value, P', of P is equal to P'. From the preceding considerations, confidence or fiducial limits of π can be chosen corresponding to any values of $\underline{n}$ and $\underline{s}$ and to any desired value, ϵ lying between 0 and 1, of P.

Consider the dependence of $P(\pi, n, s)$ upon the variable π, for any value of $\underline{s}$ other than 0 or $\underline{n}$. For values of $\underline{s}$ differing from 0 and $\underline{n}$ it is clear that $P(0, n, s)$ and $P(1, n, s)$ are zero, and that there are some values of π, between 0 and 1, for which $P(\pi, n, s)$ is unity. The values of π for which $P(\pi, n, s)$ is unity are values for which $p_{n,s}(\pi)$ is the largest of all terms $p_{n,r}(\pi)$. It can readily be shown that there are exactly $\underline{n}$ values of π between 0 to 1 at which $P(\pi, n, s)$ is discontinuous. Such discontinuities correspond to the "crossing" of $p_{n,s}(\pi)$ by other $p_{n,r}(\pi)$'s, and except at the $\underline{n}$ discontinuities $P(\pi, n, s)$ is a continuous function of π. When $\underline{s}$ is 0, the situation is somewhat different since then $P(0, n, 0)$ is unity while $P(1, n, 0)$ is zero. When $\underline{s}$ equals $\underline{n}$, $P(0, n, n)$ is zero while $P(1, n, n)$ is unity. Whatever may be the value of $\underline{s}$, there are just $\underline{n}$ values of π at which $P(\pi, n, s)$ is discontinuous.

The preceding description of some of the properties of the P-function has been included to clarify, in the reader's mind, the nature of the dependence of $P(\pi, n, s)$ upon π. To select confidence or fiducial limits,

$$p_1 \leq \pi \leq p_2$$

it is sufficient to consider the shortest interval of π that contains all the values of π for which

$$P(\pi, n, s) \geqq \epsilon, \qquad (4)$$

corresponding to any particular and possible values of $\underline{n}$ and $\underline{s}$, and to some desired value of ϵ. Corresponding to any possible values $\underline{n}$, $\underline{s}$, and ϵ it is always possible to find the shortest interval, (p_1, p_2), by calculation. The lower limit p_1 is the smallest of all π's satisfying the relation (4), and the upper limit p_2 is the largest of all π's satisfying the relation (4). Such limits p_1 and p_2 always exist, as will be shown. One may note first that when $\underline{s}$ is zero, p_1 is zero; and that when $\underline{s}$ is $\underline{n}$, p_2 is unity.

The existence of p_1 and p_2 follows rigorously from the consideration that the set of π's satisfying (4) has a lower bound, zero, and must therefore have a greatest lower bound, p_1. Similarly, the set of π's satisfying (4) has an upper bound, unity, and must therefore have a least upper bound, p_2. It can further be shown that because of the nature of the P-function and of its discontinuties, the numbers p_1 and p_2 must themselves satisfy (4).

Because of the last-mentioned property of p_1 and p_2, it follows that

$$P(\pi, n, s) < \epsilon, \qquad (5)$$

for all values of π lying outside the closed interval $p_1 \leq \pi \leq p_2$. Let us now arbitrarily choose some value of ϵ between 0 and 1 and adopt the policy, whenever in an experiment some value of $\underline{s}$ has been observed, of always asserting that

$$p_1 \leq \pi \leq p_2, \qquad (6)$$

where p_1 and p_2 correspond to the actual $\underline{n}$, the observed $\underline{s}$, and the chosen ϵ. We shall be wrong in our assertions when, and only when, π lies outside the closed interval (6), in which case $P < \epsilon$ by (5). The probability that our assertion is incorrect, which is the same as the proportion of times that we shall be wrong in such assertions in the long run, is therefore less than ϵ. Therefore the probability

that our assertion (6) is correct, which is the same as the proportion of times that we shall in the long run be right in such assertions, is greater than $1 - \epsilon$. We accordingly call p_1 and p_2 the lower and upper $1 - \epsilon$ confidence, or fiducial, limits.

In the following tables are listed confidence limits p_1 and p_2 for values of ϵ equal to .5, .34, and .10. These limits may be called the 50%, 66%, and 90% confidence limits. The tables cover all possible values of s corresponding to n = 4 and n = 8, and are sufficient for the discussion in the body of the Report to which this is the Appendix.

Confidence limits, n = 4

	50%		66%		90%	
s	p_1	p_2	p_1	p_2	p_1	p_2
0	0	.30	0	.30	0	.50
1	.16	.50	.10	.54	.026	.68
2	.29	.71	.29	.71	.14	.86
3	.50	.84	.46	.90	.32	.974
4	.70	1	.70	1	.50	1

Confidence limits, n = 8

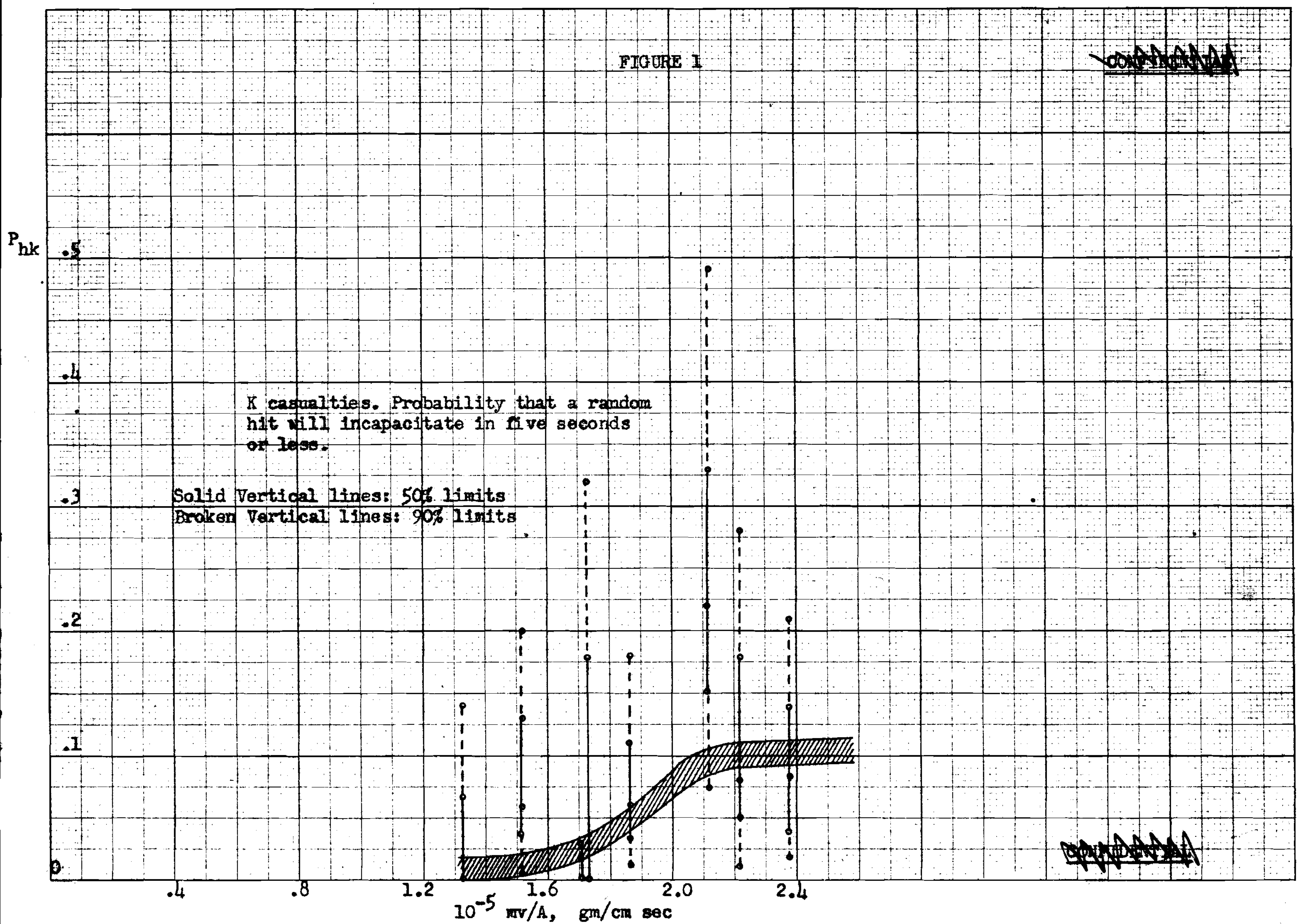
	50%		66%		90%	
s	p_1	p_2	p_1	p_2	p_1	p_2
0	0	.16	0	.21	0	.31
1	.08	.29	.05	.33	.013	.44
2	.15	.39	.15	.45	.07	.57
3	.27	.51	.20	.56	.15	.70
4	.38	.62	.32	.68	.24	.76
5	.49	.73	.44	.80	.30	.85
6	.61	.85	.55	.85	.43	.93
7	.71	.92	.67	.95	.56	.987
8	.84	1	.79	1	.69	1

The author is inclined to prefer confidence limits, selected in accordance with the procedure in this Appendix, to confidence limits selected in accordance with the procedure of Clopper and Pearson*. Our discussion has considered the probability $P(\pi, n, s)$ of obtaining a number of successes as probable as or less probable than an observed

* Clopper, C. J., and E. S. Pearson. The use of confidence or fiducial limits illustrated in the case of the binomial. Biometrika, Vol. 26, pages 404-413, 1934.

number of successes s. Their discussion considers the probability of obtaining a number of successes equal to or less than the observed number and, separately, the probability of obtaining a number of successes equal to or greater than the observed number. They select an upper confidence limit in such a way as to make the first probability less than or equal to $\epsilon/2$, and a lower limit in such a way as to make the second probability also less than or equal to $\epsilon/2$. There is, however, no value of π that can make the second probability less than or equal to $\epsilon/2$ when s is zero, nor is there any value of π that can make the first probability less than or equal to $\epsilon/2$ when s is n. Consequently, the confidence limits selected by Clopper and Pearson are unnecessarily wide when s is either 0 or n, and they have in effect over-modestly underestimated the legitimate degree of confidence in their assertions, comparable to (6), for values of s equal to 0 and n. The confidence intervals defined in this Appendix are narrower than theirs when s is 0 or n. The present intervals are not "central" in the sense in which the intervals of Clopper and Pearson are central, but they are perhaps more nearly "central" than those of Clopper and Pearson with respect to the probabilities, $P(\pi, n, s)$, of obtaining results s as probable as or less probable than some observed s, regardless of whether such less probable s's are larger or smaller than the observed s.

The practical advantage of the present procedure for selecting confidence limits, over that of Clopper and Pearson, seems appreciable in the Report to which this is an Appendix, because there s is sometimes zero.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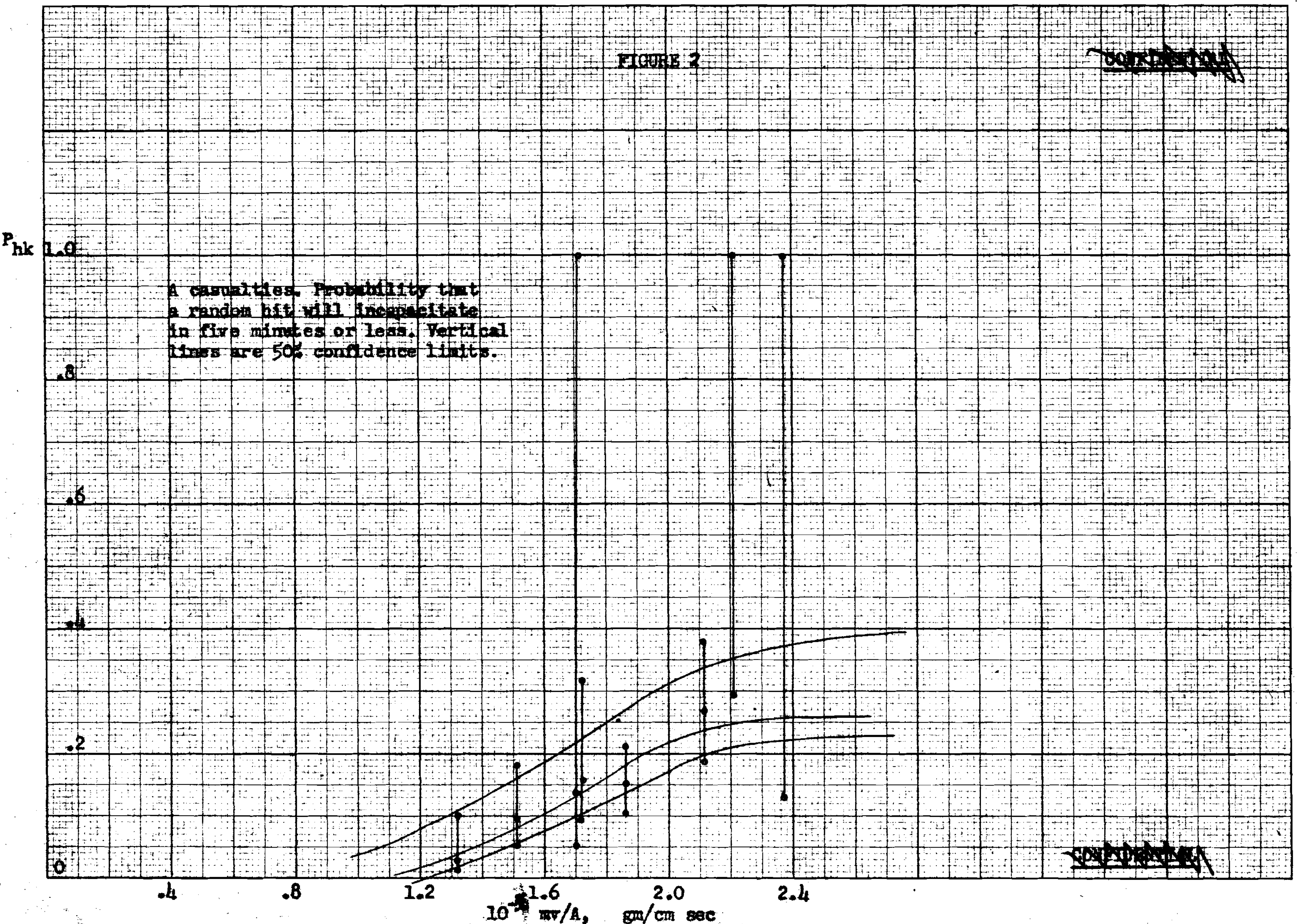
K casualties. Probability that a random hit will incapacitate in five seconds or less.
Solid Vertical lines: 50% limits
Broken Vertical lines: 90% limits
P_{hk}
.5
.4
.3
.2
.1
0
.4
.8
1.2
1.6
2.0
2.4
10^{-5} mv/A, gm/cm sec
~~CONFIDENTIAL~~

FIGURE 1

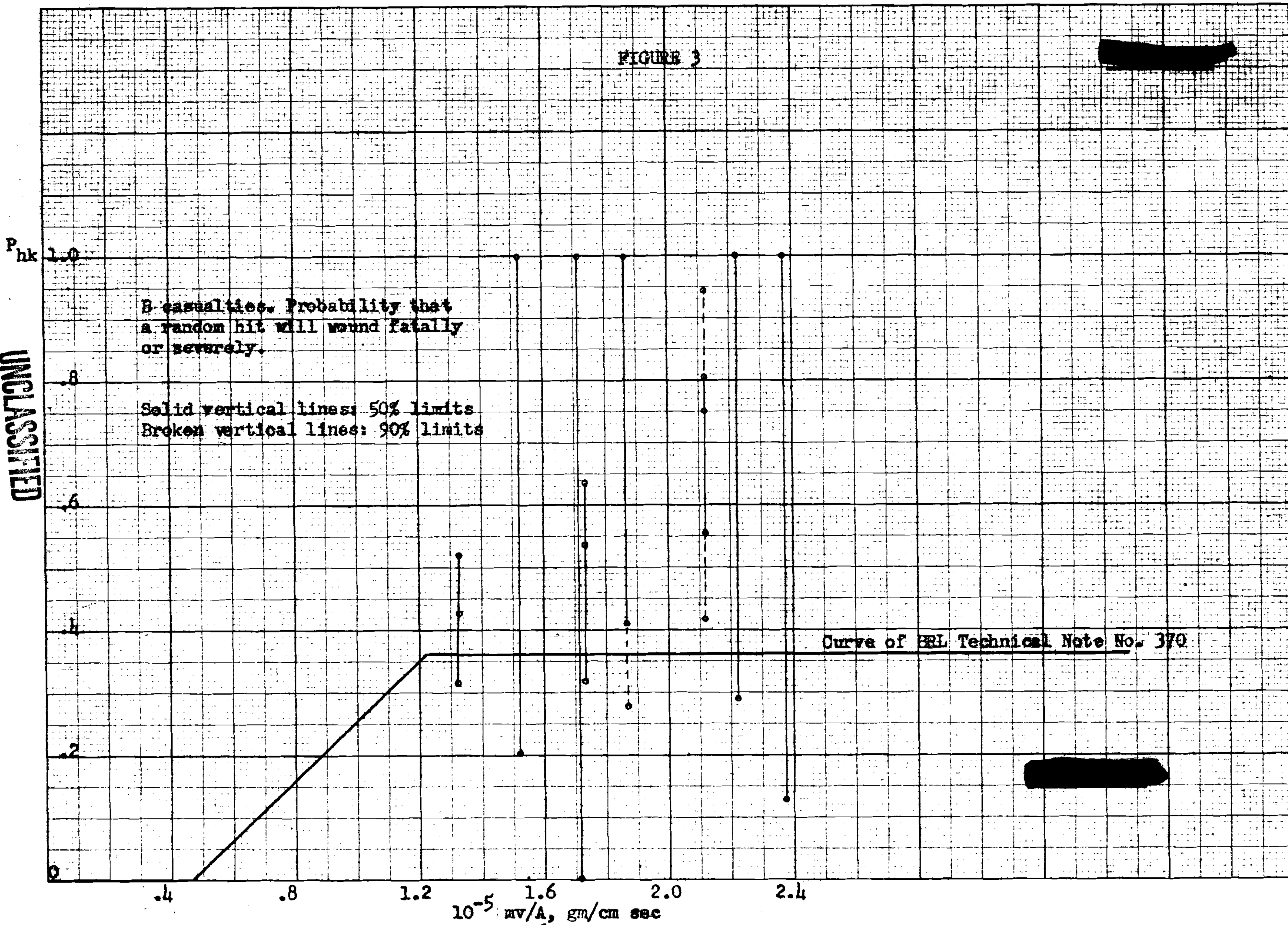
FIGURE 2



A casualties. Probability that a random hit will incapacitate in five minutes or less. Vertical lines are 50% confidence limits.

FIGURE 3

[redacted]



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